Reliability Testing for Small Sample Censored and Missing Data with Applications in Engineering

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Outline of the talk

1. A generalized gamma distribution and its properties

2. The exact likelihood ratio test of the homogeneity

3. The exact likelihood ratio test of the scale

4. Applications and illustrative examples (missing and censored data, mixtures)
A generalized gamma distribution Let us consider a sample from a generalized gamma distribution (ggd) density, introduced by Stacy (1962), of the form

\[ f(y_i|\vartheta_i) = \frac{\alpha}{\sigma_i \Gamma\left(\frac{1+\beta}{\alpha}\right)} \left(\frac{y_i}{\sigma_i}\right)^{\beta} \exp\left(-\left(\frac{y_i}{\sigma_i}\right)^{\alpha}\right), \quad y_i > 0, \]

and \( \vartheta_i = (\alpha, \beta, \sigma_i) \), where \( \alpha, \beta > 0 \) are shape parameters and \( \sigma_i > 0 \) is a scale parameter. Ggd has many applications in reliability, engineering, physics, finance, e.g.

fitting the Compton shots in Astrophysics (see Nowak et al., 1999)

for applications in hydrology see Hayakawa, Irony, and Xie (2001)
A special cases of generalized gamma

The generalized gamma distribution is one of the most studied probability density functions of statistics since many of the important density functions can be derived from it. For example,

(1) $f(y|(2, 0, \sqrt{2\sigma}))$ is the one-sided normal distribution,

(2) $f(y|(1, n/2 - 1, 2))$ is the $\chi^2_n$-distribution.

(3) For $\beta = \alpha - 1$ the generalized gamma is a Weibull distribution

(4) in case of $\alpha = 1$ we obtain the gamma distribution.

(5) lognormal distribution is the limit as $\nu \to \infty$
Here we provide an exact LR test of the **homogeneity hypothesis**

\[ H_0 : \sigma_1 = \ldots = \sigma_N \]  \hspace{1cm} (1)

versus \( nonH_0 \).

Under the homogeneity we provide the exact LR test of the **scale hypothesis**

\[ H_0 : \sigma = \sigma_0 \text{ versus } H_1 : \sigma \neq \sigma_0. \]  \hspace{1cm} (2)
Homogeneity testing (Theorem 1 in Stehlík 2006)

Let \(y_1, \ldots, y_N\) be iid according to generalized gamma distribution with the unknown common scale parameter \(\sigma\). Then the LR homogeneity statistics \(-\ln \Lambda_N\) is

\[
N\left(\frac{\beta + 1}{\alpha}\right) \ln\left(\sum_{i=1}^{N} y_i^\alpha\right) - N\left(\frac{\beta + 1}{\alpha}\right) \ln N - \left(\frac{\beta + 1}{\alpha}\right) \sum_{i=1}^{N} \ln(y_i^\alpha).
\]

It has the same distribution as the random variable

\[-\left(\frac{\beta + 1}{\alpha}\right) \ln\{N^N u_1 \ldots u_{N-1}(1 - u_1 - \ldots - u_{N-1})\},\]

where the vector \((u_1, \ldots, u_{N-1})\) has a generalized Beta distribution \(B\left(\frac{\beta + 1}{\alpha}, \ldots, \frac{\beta + 1}{\alpha}\right)\) on the simplex

\[\{u : 0 < u_1 < 1, \ldots, 0 < u_{N-1} < 1 - u_1 - \ldots - u_{N-2}\}.
\]
Some notes on homogeneity testing

The generalized Beta distribution is in the literature also called the Dirichlet distribution or the multivariate Beta distribution. We can use a well-implemented random number generator of Dirichlet distribution in R, Matlab,.. to generate a critical values of the test.

The distribution of LR is under the homogeneity independent on unknown scale parameter of sample (this is an advantage against some asymptotical tests and tests dependent on true but unknown value of $\sigma$)
Scale testing (Theorem 2 in Stehlík 2006)

The exact cumulative distribution function of the Wilks statistics $-2 \ln \Lambda = 2 G_N (\sum_{i=1}^N \left( \frac{y_i}{\sigma_0} \right)^\alpha) - 2 G_N (N)$ of the LR test of the hypothesis

\[ H_0 : \sigma = \sigma_0 \text{ versus } H_1 : \sigma \neq \sigma_0 \]

has under $H_0$ the form

\[
F_{vN}^\Gamma \left\{ -vNW_{-1} \left( -e^{-1-\frac{\tau}{2vN}} \right) \right\} + \\
-F_{vN}^\Gamma \left\{ -vNW_0 \left( -e^{-1-\frac{\tau}{2vN}} \right) \right\}, \quad \tau > 0
\]

where $v = \frac{\beta+1}{\alpha}$. The Lambert W function is defined to be the multivalued inverse of the complex function $f(y) = ye^y$. As the equation $ye^y = z$ has an infinite number of solutions for each (non-zero) value of $z \in \mathbb{C}$, the Lambert W has an infinite number of branches (see Corless, 1996).
Consider a testing problem $H_0 : \vartheta \in \Theta_0$ vs $H_1 : \vartheta \in \Theta_1 \setminus \Theta_0$, where $\Theta_0 \subset \Theta_1 \subset \Theta$. Further consider sequence $T = \{T_N\}$ of test statistics based on $y_1, ..., y_N$ iid $\sim P_\vartheta, \vartheta \in \Theta$. We reject for large values of test statistics.

For $\vartheta$ and $t$ denote $F_N(t, \vartheta) := P_\vartheta\{s : T_N(s) < t\}; \quad G_N(t) := \inf\{F_N(t, \vartheta) : \vartheta \in \Theta_0\}$. The quantity $L_n(s) = 1 - G_n(T_n(s))$ is called the attained level or the $p$-value. Suppose that for every $\vartheta \in \Theta_1$ the equality

$$\lim_{n} \frac{-2 \ln L_n}{n} = c_T(\vartheta)$$

holds a.e. $P_\vartheta$. Then the nonrandom function $c_T$ defined on $\Theta_1$ is called the Bahadur exact slope of the sequence $T = \{T_n\}$. 

Raghavachari (1970) and Bahadur (1971) have proved

\[ c_T(\vartheta) \leq 2I(\vartheta, \Theta_0) \] (3)

holds for each \( \vartheta \in \Theta_1 \). Here \( I(\vartheta, \Theta_0) := \inf \{ I(\vartheta, \vartheta_0) : \vartheta_0 \in \Theta_0 \} \).

If (3) holds with the equality sign for all \( \vartheta \in \Theta_1 \), then the sequence \( T \) is said to be asymptotically optimal in the Bahadur sense.

The maximization of \( c_T(\vartheta) \) is a nice statistical property.
The asymptotic cumulative distribution function of the Wilks statistics $-2 \ln \Lambda$ of the LR test of the hypothesis

$$H_0 : \sigma = \sigma_0 \text{ versus } H_1 : \sigma \neq \sigma_0$$

has under $H_0$ $\chi^2_1$ distribution (Wilks, 1963)
I-divergence: the connection between tests

Let $y_i$ are iid $\Gamma(v_i, \gamma_i)$ We define the $I$-divergence of the observed vector $y$ in the sense of Pázman (1993) as

$$I_N(y, \gamma) := I(\hat{\gamma}y, \gamma) = - \sum_{i=1}^{N} \{v_i - v_i \ln(v_i)\} + \sum_{i=1}^{N} \{y_i \gamma_i - v_i \ln(y_i \gamma_i)\}.$$ 

In Stehlík (2003) is proved that in distribution

$$I_N(y, \gamma(0)) = - \ln \Lambda_s \oplus (- \ln \Lambda_H | \gamma_1 = \ldots \gamma_N)$$

where $\gamma_i = 1/\sigma_i$, $\gamma(0) = (\gamma_0, \ldots, \gamma_0)$. 

The oversizing of the asymptotical scale test

The oversizing of the asymptotical test can be defined as the difference between $\alpha_{e,N}$ and $\alpha$, where $\alpha_{e,N} = 1 - F_N(\chi^2_{\alpha,1})$ and $\chi^2_{\alpha,1}$ denotes $(1-\alpha)$-quantile of the asymptotical $\chi^2_1$-distribution, $F_N$ is the exact cdf of the Wilks statistics $-2 \ln \Lambda$ under the $H_0$.

Here $\alpha$ is the proposed size of the test while $\alpha_{e,N}$ is the obtained size of the test.

The table giving the oversizing of the asymptotical test when the observations are distributed exponentially is following.
Table 1: The exact sizes $\alpha_{e,N}$

<table>
<thead>
<tr>
<th>$\alpha \setminus N$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<td>0.178154e-4</td>
<td>0.155230e-4</td>
<td>0.142351e-4</td>
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<td>0.1439040e-3</td>
<td>0.1334225e-3</td>
<td>0.1268967e-3</td>
</tr>
<tr>
<td>0.0002</td>
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<td>0.2808525e-3</td>
<td>0.2614285e-3</td>
<td>0.2493896e-3</td>
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<tr>
<td>0.0005</td>
<td>0.7598856e-3</td>
<td>0.6789730e-3</td>
<td>0.6356518e-3</td>
<td>0.6089504e-3</td>
</tr>
<tr>
<td>0.001</td>
<td>0.14706397e-2</td>
<td>0.13227325e-2</td>
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<td>0.11960162e-2</td>
</tr>
<tr>
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<td>$\alpha \backslash N$</td>
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<td>30</td>
<td>40</td>
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Oversizing and inappropriate rejection

In other words, the oversizing of the asymptotical test means, that for critical constants the inequality holds

\[ \chi^2_{\alpha,1} < c_{N,\alpha}, \]

where \( c_{N,\alpha} \) is the critical constant of the \( \alpha \)-sized exact test based on the Wilks statistics when the sample size is \( N \).

It means that for all \( \sigma_0 \) such that

\[ \chi^2_{\alpha,1} < -2 \ln \Lambda \leq c_{N,\alpha} \]

we reject the null hypothesis \( H_0 : \sigma = \sigma_0 \) statistically incorrectly (by inappropriate use of \( \chi^2_1 \)-asymptotics).
Illustrative examples
Example 1: exact LR inference for censored data

Progressive Type-II censoring is the versatile scheme of censoring studied thoroughly by Balakrishnan and Aggarwala (2000):

From a total of $N$ units placed on a life-test only $m$ are completely observed until failure. At the time of the first failure, $R_1$ of the $n-1$ surviving units are randomly withdrawn (or censored) from the life-testing experiment. At the time of the next failure, $R_2$ of the $n-2-R_1$ surviving units are randomly withdrawn (or censored), and so on. Finally, at the time of the $m$-th failure, all the remaining $R_m = n - m - R_1 - \ldots - R_{m-1}$ surviving units are censored. Censoring takes place here progressively in $m$ stages. This scheme includes as special cases the complete sample situation (when $m = n$ and $R_1 = R_1 = \ldots = R_m = 0$) and the conventional Type II right censoring situation (when $R_1 = \ldots = R_{m-1} = 0$ and $R_m = n - m$).
The inference for the one parameter (scale) exponential (what corresponds to the one parameter Weibull with the known shape parameter) for the progressively Type II Right \((R_1, \ldots, R_m)\) censored sample is equivalent with the inference based on the complete sample with size \(m\) (see also Viveros and Balakrishnan (1994) for some estimation aspects). This conclusion for the exact testing can easily be seen from the following: The log-likelihood ratio statistics of the test of hypothesis (1) under progressive right Type II censoring has the form

\[
-\ln \Lambda = G_m \left( \sum_{i=1}^{m} (R_i + 1) \left( \frac{x_i \cdot m \cdot N}{\sigma_0} \right)^\alpha \right) - G_m(m),
\]

where \(G_m(x) = x - m \ln x, x > 0\) is the function introduced in Stehlík (2003).
We have (Balakrishnan and Stehlík (2008))

\[ F(\tau) = H \left( -mW_{-1}(e^{-1-\frac{\tau}{2m}}) \right) - H \left( -mW_{0}(e^{-1-\frac{\tau}{2m}}) \right), \quad \tau > 0, \]

where \( H \) is the cumulative distribution function of the random variable

\[ Y = \sum_{i=1}^{m} (R_i + 1) \left( \frac{x_{i:m:N}}{\sigma_0} \right)^\alpha. \]

By the power transformation (since the shape parameter is assumed to be known) everything can be transformed to the exponential case.

\[ H = F_{m}^{\Gamma} \] since the random variable \( Y \) can be written as a sum of progressively Type II right censored spacings which are independent exponential (see Thomas and Wilson, 1972).
**Progressively Type II right censored spacings**

Let $X_{i:m:N}, i = 1, \ldots, m$ be a progressively Type II right censored sample from Exp(1). The Progressively Type II right censored spacings are defined as

$$Z_1 = NX_{1:m:N},$$

$$Z_2 = (N - R_1 - 1)(X_{2:m:N} - X_{1:m:N}),$$

$$\ldots$$

$$Z_m = (N - R_1 - \ldots - R_{m-1} - m + 1)(X_{m:m:N} - X_{m-1:m:N})$$

and they are iid Exp(1) (see Thomas and Wilson, 1972).

For a special case of no censoring ($R_1 = \ldots = R_m = 0$) we obtained the spacings introduced by Sukhatme (1937).
The Type I censoring is much more complex. The difference from the testing point of view is (Balakrishnan and Stehlík (2008)):

1) in progressively Type II censoring we have a pivotal statistics, 
\[ T_n = \sum_{i=1}^{m} \frac{S_i}{\sigma_0} \sim \Gamma(m, 1) \]
We can use the ”naive” \( T_n \) or ELRT statistics

2) in Type I censoring we have no pivotal statistics, so it is natural to use the ELRT statistics
Comparison of "naive" pivot $T_n$ and ELRT

**Example** (Balakrishnan and Stehlík (2008)) Type II censoring, $\sigma_0 = 1, m = 10$, shape $\alpha = 1$ size of the test is 0.0266 Power functions are

$$p_c(\sigma) = 1 - F^{\Gamma}_{10}(-10\sigma W_1(-e^{-1.25})) + F^{\Gamma}_{10}(-10\sigma W_0(-e^{-1.25}))$$

$$p_{T_n}(\sigma) = 1 - F^{\Gamma}_{10}(18.5/\sigma) + F^{\Gamma}_{10}(4.3/\sigma)$$
red=ELRT, blue=T_n
**ELRT for Type I censoring**

In Type I censoring we have no pivotal statistics, so it is natural to use the ELRT statistics

a) Conditioning with the number of observed failures $m$. The interpretation and usage of ELRT is different, since you force to observe a fixed number of failures in the given interval in future.

b) The "unconditional" (it is always conditioned by "at least one failure occurs") likelihood inference is the joint research with Balakrishnan (2007)
The cdf of the Wilks LR statistics

\[-2 \ln \Lambda = 2G_m \left( \sum_{i=1}^{m} \left( \frac{x_i}{\sigma_0} \right)^{\alpha} + (N - m) \left( \frac{T}{\sigma_0} \right)^{\alpha} \right) - 2G_m(m)\]

of the scale hypothesis (1) has the form \(\tau > 0\) (Balakrishnan and Stehlík (2008))

\[F(\tau) = \sum_{m=1}^{N} \left[ H_m \left( -mW_{-1}(-e^{-1-\frac{\tau}{2m}}) \right) - H_m \left( -mW_0(-e^{-1-\frac{\tau}{2m}}) \right) \right] p(m),\]

where \(H_m\) is the cdf of the random variable

\[Y_m = \sum_{i=1}^{m} \left( \frac{x_i}{\sigma_0} \right)^{\alpha} + (N - m) \left( \frac{T}{\sigma_0} \right)^{\alpha}.\]

Here \(p(m)\) is truncated binomial distribution \(b(N, 1 - \exp(-\frac{T}{\sigma_0}))\) excluding 0,
Notice (see Balakrishnan and Stehlík (2008))

a) $H_m$ is the CDF of the random variable $Y_m$ which can be adjusted from Childs et al. (2003) by finding the distribution of $m\sigma$ under the condition that $m$ units are completely observed until failure. In contrary to Type II censoring case $Y_m$ has a finite support since $Y_m \leq N\left(\frac{T}{\sigma_0}\right)^\alpha$.

b) $h_m$ is not a pure mixture but a so called generalized mixture, since summand signs can alternate.
Example Let us consider the exponential data given by Bartholomew (1963) and later elaborated by Childs et al. (2003) in the Type-II hybrid censoring scheme.

The data are consisting of $N = 20$ items being put on a life test for a prefixed time of 150 hours resulting in the following observed failure times:

$$3, 19, 23, 26, 27, 37, 38, 41, 45, 58, 84, 90, 99, 109, 138.$$

In order to illustrate ELRT, we suppose that a censoring time of $T = 50$ was used, and we use for the null hypothesis the value of $\sigma_0 = 101.8$, given by Childs et al. (2003).

Thus we consider the testing problem

$$H_0 : \sigma = 101.8 \text{ versus } H_1 : \sigma \neq 101.8.$$
The Wilks statistics has the value (Balakrishnan and Stehlík (2008)) \(-2 \ln \Lambda = 2G_9(7.9469) - 2G_9(9) = 33.29827857\). The exact power function \(p(\sigma)\) of the hypothesis (1) for a observed data has the form

\[
1 - \sum_{m=1}^{20} \left[ H_m \left( -mW_{-1} \left( -e^{-1 - \frac{16.649}{m}} \right) \right) - H_m \left( -mW_0 \left( -e^{-1 - \frac{16.649}{m}} \right) \right) \right] p(m),
\]

where \(H_m\) and \(p_m\) are computed at the given value of the alternative \(\sigma\).
Example 2: The analysis of photoemulsion experiment.

(This work has been supported by LIT JINR, Dubna) see Stehlík and Ososkov (2003) for details.

The physical background: the emulsion experiment studying the dynamics of inelastic collision of fast heavy particles as nuclei $^{22}\text{Ne}$ with the photoemulsion nuclei by momenta $4.1\text{GeV}/c$.

$$f(y; P_{\perp}) = \sum_{i=1}^{k} a_i \frac{y}{\sigma_i^2} \exp\left(-\frac{y^2}{2\sigma_i^2}\right), y > 0, \sum a_i = 1$$

with some unknown $k, \sigma_i$, and $a_i$.

Typically in such experiments the number of observations is small and asymptotical procedures misleads.
Homogeneity test

The widely used approach (for instance see chapter 19.4 in Balakrishnan and Basu (1996) among others) is to use a LRT statistic $2 \ln \Lambda = 2(L(\hat{\vartheta}_1) - L(\hat{\vartheta}_0))$ where $\hat{\vartheta}_0$ and $\hat{\vartheta}_1$ are the MLE’s of the scales under the null and alternative. This plug-in LRT parameter estimation is usually accomplished by the EM algorithm, however the calculation of the test statistic and the Monte-Carlo simulation of its null distribution depend heavily on the particular implementation of the EM algorithm, e.g. starting and stopping strategies (see Seidel et al. (2000)).

But this is definitely not needed in exponential case (despite the normal one) since we have a pivotal LR test statistics with known exact distribution! (see Stehlík (2003) and (2006)) This test is even AOBS (see Rublík, 1989a and 1989b and Stehlík, 2003).
And pivotal statistics exists even when the alternative fix the number of components (see Stehlík and Ososkov, 2003):

Let $H_1 : m = 2$ and $y_1, ..., y_N$ are iid according to the Rayleigh distribution with the unknown scale parameter $\sigma$. Then

$$-\ln \Lambda_N = -N \ln N - \min_{0 < K < N} \{-K \ln K - (N - K) \ln (N - K) +$$

$$+ K \ln \left( \sum_{i=1}^{K} y_{(i)}^2 \right) + (N - K) \ln \left( \sum_{i=K+1}^{N} y_{(i)}^2 \right) \}$$

and it has the same distribution as the random variable

$$U_N = -\min_{0 < K < N, p \in P(K)} \{N \ln N - K \ln K - (N - K) \ln (N - K) +$$

$$+ K \ln \left( \sum_{n=1}^{K} u_{in} \right) + (N - K) \ln \left( \sum_{n=1}^{N-K} u_{in} \right) - N \ln \left( \sum_{n=1}^{N} u_n \right) \}$$

where $u_1, ..., u_N$ are iid according to $Exp(1)$. 
Competitive risk alternatives (see Stehlík and Wagner, 2008):

$$H_0 : y_1, \ldots, y_N \sim \theta \text{ against } H_0 : y_1, \ldots, y_N \sim \pi \exp(-y) + (1 - \pi)\theta \exp(-\theta y)$$

We generated $nsim = 10000$ samples of $n = 100, 1000$

$\theta = 1, 1.25, 1.5, 2, 3, 4, 5, 7, 10$ and $\pi = 0.1, 0.9$.

Results for $\pi = 0.1$ (upper contamination), for $\pi = 0.9$ (lower contamination):

We have observed that the ELRTs are best for the lower contamination (in the sense of Mosler) but not for the upper contamination.
Example 3 (Real data): analyzing of the reliability changes (see Stehlík 2006,b)

The real data is the airplane indicator light operating time from RAC database (see Coit and Jin, 2000).

**Table 1:** Airplane indicator light reliability data

<table>
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<tr>
<th>Failures</th>
<th>$T_j$</th>
<th>Cumulative operating time (hours)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$T_1$</td>
<td>51 000</td>
</tr>
<tr>
<td>9</td>
<td>$T_2$</td>
<td>194 900</td>
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<td>8</td>
<td>$T_3$</td>
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<td>8</td>
<td>$T_4$</td>
<td>112 400</td>
</tr>
<tr>
<td>6</td>
<td>$T_5$</td>
<td>104 000</td>
</tr>
<tr>
<td>5</td>
<td>$T_6$</td>
<td>44 800</td>
</tr>
</tbody>
</table>
In Coit and Jin (2000) we can found the MLE of shape parameter (0.7) and scale parameter (0.0000484) of the gamma distributed individual times-to-failure of the data in Table 1.

We have \( \omega = 38 \times 0.7 = 26.6 \) and \( \sum_{j=1}^{6} T_j = 552,400 \).

Let us consider the testing problem

\[
H_0 : \sigma = 0.00003207 \quad \text{versus} \quad H_1 : \sigma \neq 0.00003207 \quad (4)
\]

at the level of significance 0.05.

In particular a reliability practitioner could be interested in conducting the hypothesis (4) test, to see whether the field reliability has significantly changed from its current level.
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References


Balakrishnan N. and Stehlík M. (2008), Exact likelihood ratio test of the scale for censored Weibull sample, IFAS Res. Report Nr. 35 (submitted)


Moran P.A.P. (1951) The random division of an Interval II, JRSS B, 13, 147-50


Stehlík M. (2007). Exact testing of the scale with the missing time-to-failure information, Communications on Dependability and Quality Management in Engineering (CDQM), 10(2): 124-129.
Stehlík M. (2008,a) Exact testing of the scale parameter with the missing time-to-failure information, under revision for *ANZJS*.


Stehlík M. and Wagner H. (2008,c), Exact likelihood ratio testing of homogeneity for the exponential distribution, IFAS Research Report

Sukhatme P.V. (1937). Tests of significance for samples of the $\chi^2$ population with two degrees of freedom, Annals of Eugenics, 8, 52-56.


